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THE APPLICATION OF THE POWER SERIES TRANSFORM TO LINEAR DIFFERENCE EQUATIONS IN ENGINEERING

by Leonard McFadden

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The Application of the Power Series Transform to Linear Difference Equations in Engineering

by
Leonard McFadden(1)

Introduction

Certain problems in engineering involve linear difference equations, usually of the second order. The purpose of this paper is to show the application of the power series transform to the solutions of such problems. The examples selected to illustrate the method include problems from the field of Civil Engineering such as, continuous beams, rigid frames, vibrating shafts, etc. It is well known that these problems can be solved by other methods such as moment distribution, slope deflection, and the method of Stodola, but the method to be presented here has the advantage in certain cases.

The method of solving linear difference equations by the Laplace Transform appears in literature(2), but the method of the power series transform has the advantages that (a) less mathematical background is required, (b) the formulas involved are less cumbersome, (c) the formulas are more easily established, and (d) the algebraic manipulation is less complicated.

The Power Series Transform

Given a sequence $\{f_k\}$, $k = 0,1,2 - - -$, its power series transform is defined as $P(f_k) = \bar{f}(s) = \sum_0^\infty f_k s^k$. Thus to every sequence there corresponds a power series in which the coefficient of the k'th power of s is the k'th term of the sequence. This series is the transform

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of the given sequence. In certain cases it is possible to recogonize $\bar{f}(s)$ as the Maclaurin expansion of a known function for those values of s for which the series converges. Thus for example if $f_k = 1$, $k = 0,1,2, - - -$ then $\bar{f}(s) = \sum_0^\infty s^k = \frac{1}{1-s}$, for $-1 < s < 1$.

Appendix I contains a short list of properties of the transform, and the transforms of certain functions which will prove useful in the illustrative problems. A few of these formulas are established in Appendix II to illustrate the general procedure.

Solution of Difference Equations

The difference equations considered will be written in the form $f_{k+2} + 2b\, f_{k+1} + c\, f_k = F(k)$, where $F(k)$ is a known function of k. The result of transforming this equation term by term, using formulas 1 and 2 of Appendix I on the left hand side, and the appropriate formula or formulas from the same list on the right hand side, is

$$\frac{\bar{f}(s)}{s^2} - \frac{1}{s^2}(f_o + s\, f_1) + 2b\left\{\frac{\bar{f}(s)}{s} - \frac{1}{s} f_o\right\} + c\, \bar{f}(s) = \bar{F}(s),$$

$$\text{or } \left(\frac{1}{s^2} + \frac{2b}{s} + c\right)\bar{f}(s) = \bar{F}(s) + \frac{1}{s^2}(f_o + s\, f_1) + \frac{2b}{s} f_o.$$

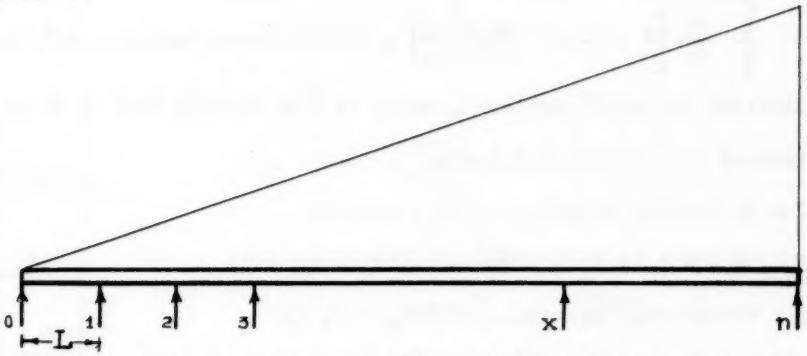
On dividing through by the coefficient of $\bar{f}(s)$ this becomes

$$\bar{f}(s) = \frac{s^2 \bar{F}(s)}{1+2bs+cs^2} + \frac{f_o + (f_1+2bf_o)s}{1 + 2bs + c\, s^2}.$$

The inverse transforms of the terms on the right hand side of this equation can now be found by reference to the list. Note that if the values of f_o and f_1 are known then the problem is completely solved. If only f_o is known it is necessary to impose another boundary condition to determine f_1, and if neither f_o nor f_1 is known it is necessary to impose two known boundary conditions to determine their values.

Illustrative Examples

1. Consider a continuous beam with n equal spans and a triangular load distribution, the total load being P. Find the moment M_x at the x'th support.



The relation between the moments at any three adjacent supports can be expressed by the three moment equation as follows:

$M_{x+2} + 4\,M_{x+1} + M_x = -\frac{PL}{n^2}(x+1)$, and the moments at the end supports are $M_o = M_n = 0$.

The transformed equation, with $M_o = 0$ is

$$\left(\frac{1}{s^2} + \frac{4}{s} + 1\right)\bar{M} = -\frac{PL}{n^2}\left(\frac{s}{(1-s)^2} + \frac{1}{1-s}\right) + \frac{M_1}{s},$$

or

$$\bar{M} = -\frac{PL}{n^2}\,\frac{s^2}{(1-s)^2(1+4s+s^2)} + \frac{M_1 s}{1+4s+s^2},$$

which becomes by use of partial fractions

$$\bar{M} = -\frac{PL}{6n^2}\left[\frac{s}{(1-s)^2} - \left(\frac{6M_1 n^2}{PL} + 1\right)\frac{s}{1+4s+s^2}\right]$$

The result of taking the inverse transformation and setting $\frac{6M_1 n^2}{PL} + 1 = K$ is

$$M_x = -\frac{PL}{6n^2}\left[x - K(-1)^{x-1}\,\frac{\sinh \lambda x}{\sinh \lambda}\right], \text{ where } \cosh \lambda = 2$$

Since $M_n = 0$ it follows that $K = (-1)^{n-1}\, n\, \frac{\sinh \lambda}{\sinh \lambda n}$.

Therefore the moment at the x'th support is given by

$$M_x = -\frac{PL}{6n}\left[\frac{x}{n} - (-1)^{x+n}\ \frac{\sinh\lambda x}{\sinh\lambda n}\right]$$

$$\begin{cases} -\frac{PL}{6n}\left[\frac{x}{n} + (-1)^{x}\ \frac{\sinh\lambda x}{\sinh\lambda n}\right], & \text{if n is odd} \\ -\frac{PL}{6n}\left[\frac{x}{n} - (-1)^{x}\ \frac{\sinh\lambda x}{\sinh\lambda n}\right], & \text{if n is even, where } \cosh\lambda = 2. \end{cases}$$

2. Consider the same problem as above with a uniform load of w lbs per foot instead of a triangular load.

The difference equation in this case is

$$M_{x+2} + 4\,M_{x+1} + M_x = -\frac{wL^2}{2} \text{ where } M_0 = M_n = 0.$$

The transformed equation, with $M_0 = 0$, is

$$\left(\frac{1}{s^2} + \frac{4}{s} + 1\right)\ \bar{M} = -\frac{wL^2}{2}\,\frac{1}{1-s} + \frac{M_1}{s},$$

or $$\bar{M} = -\frac{wL^2}{2}\,\frac{s^2}{(1-s)(1+4s+s^2)} + \frac{M_1\,s}{1+4s+s^2}$$

$$= -\frac{wL^2}{12}\left[\frac{1}{1-s} - \frac{5s+1}{1+4s+s^2}\right] + \frac{M_1\,s}{1+4s+s^2}$$

$$= -\frac{wL^2}{12}\left[\frac{1}{1-s} - \frac{1+2s}{1+4s+s^2} - K\,\frac{s}{1+4s+s^2}\right], \text{ where } K = 3 + \frac{12M_1}{wL^2}.$$

The inverse transformation gives

$$M_x = -\frac{wL^2}{12}\left[1-(-1)^x\ \cosh\lambda x - (-1)^x\,K\,\sinh\lambda x\right], \text{ where } \cosh\lambda = 2.$$

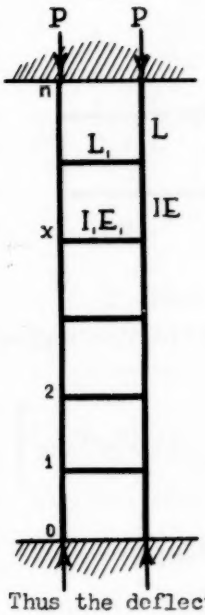
Since $M_n = 0$, then $K = \frac{1-(-1)^n\cosh\lambda n}{(-1)^n\sinh\lambda n}$.

Therefore $$M_x = -\frac{wL^2}{12}\left[1 - (-1)^x\cosh\lambda x - (-1)^{n+x}\left\{1-(-1)^n\cosh\lambda n\right\}\frac{\sinh\lambda x}{\sinh\lambda n}\right]$$

$$= \begin{cases} -\frac{wL^2}{12}\left[1 - (-1)^x\ \frac{\cosh\left(\frac{n}{2} - x\right)\lambda}{\cosh\frac{n\lambda}{2}}\right], & \text{for n even} \\ -\frac{wL^2}{12}\left[1 - (-1)^x\ \frac{\sinh\left(\frac{n}{2} - x\right)\lambda}{\sinh\frac{n\lambda}{2}}\right], & \text{for n odd, where } \cosh\lambda = 2. \end{cases}$$

3. A rectangular truss consisting of two vertical columns fixed at their ends, and a number of cross beams form n identical frames. An axial load P is applied to each column. If it is assumed that the buckling is due to shear only, the relation between the slopes of the columns at any three consecutive joints is[3]

$$v_{x+2} - 2\left(\cos\alpha L + 3\mu \frac{\sin\alpha L}{\alpha L}\right) v_{x+1} + v_x = 0, \text{ where } v_o = v_n = 0,$$



and where $\alpha = \sqrt{\frac{P}{E\,I}}$ and $\mu = \frac{E_1\,I_1\,L}{E\,I\,L_1}$

On taking the transform of this equation and setting $b = \cos\alpha L + 3\mu \frac{\sin\alpha L}{\alpha L}$ there results

$$\left(\frac{1}{s^2} - \frac{2b}{s} + 1\right)\bar{v} = \frac{v_1}{s}, \text{ or } \bar{v} = \frac{v_1\, s}{1-2bs+s^2}$$

Hence from the inverse transform,

$$v_x = v_1 \frac{\sin\lambda x}{\sin\lambda}, \text{ where } \cos\lambda = b \text{ if } b^2 < 1.$$

Since $v_n = 0$, then $v_1 \frac{\sin\lambda n}{\sin\lambda} = 0$. Therefore

$$\lambda n = k\pi \text{ or } \lambda = \frac{k\pi}{n}, k = 0, 1, 2, 3 - - -$$

Thus the deflection modes are given by

$$v_x = v_1 \frac{\sin\frac{k\pi x}{n}}{\sin\frac{k\pi}{n}} \text{ where } \cos\frac{k\pi}{n} = \cos\alpha L + 3\mu\frac{\sin\alpha L}{\alpha L}.$$

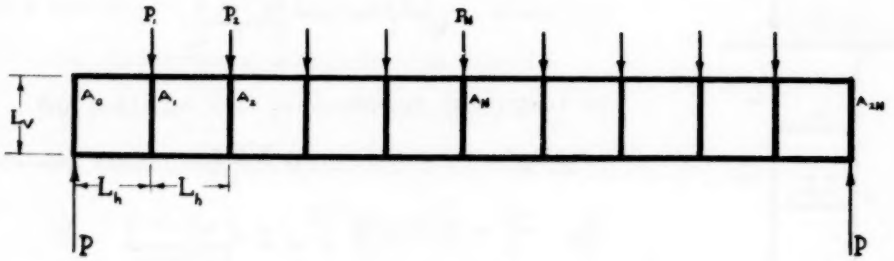
If $k = 0$ there is no buckling and if $k = 1$ there are no vertical tangents except at the ends, hence the critical deflection mode is given by

$$v_x = v_1 \frac{\sin\frac{2\pi x}{n}}{\sin\frac{2\pi}{n}}$$

and the critical buckling load can be determined from the equation

$$\cos\frac{2\pi}{n} = \cos\alpha L + 3\mu\frac{\sin\alpha L}{\alpha L}.$$

4. Consider a Vierendeel Truss with parallel chords where the stiffness of the horizontal members is K_h and that of the vertical members is K_v. Let the truss consist of 2N identical panels of length L_h and depth L_v, and the loads P_i be applied at the panel points, symmetric to the midpoint.



The difference equation satisfied by three consecutive deflection angles A is[4]

$$\left(\frac{1}{4}K_h + \frac{3}{2}K_v + \frac{1}{4}K_h\right) A_{n+1} - \frac{1}{4}K_h A_n - \frac{1}{4}K_h A_{n+2} = \frac{L_h}{4}\left[2\sum_{i=n+1}^{N-1} P_i + (P_N - P_{n+1})\right]$$

$$\text{or } A_{n+2} - 2\left(1+3\frac{K_v}{K_h}\right) A_{n+1} + A_n = -\frac{L_h}{K_h}\left[2\sum_{n+1}^{N-1} P_i + (P_N - P_{n+1})\right]$$

$$\text{where } A_N = 0 \text{ and } A_1 - \left(1+6\frac{K_v}{K_h}\right) A_o = -\frac{L_h}{K_h}\left[\sum_{i=1}^{N-1} P_i + \frac{P_N}{2}\right].$$

If in particular $P_1 = P_2 = \cdots = P_N = P$ this equation becomes

$$A_{n+2} - 2b\, A_{n+1} + A_n = -2\frac{L_h P}{K_h}(N-n-1)$$

$$\text{where } A_n = 0,\ A_1 - (2b-1) A_o = -\frac{L_h P}{K_h}(N - 1/2) \text{ and } b = 1+3\frac{K_v}{K_h}.$$

The transformed equation is

$$\left(\frac{1}{s^2} - \frac{2b}{s} + 1\right)\bar{A} = \frac{A_o + A_1 s}{s^2} - \frac{2bA_o}{s} - \frac{2L_h P}{K_h}\frac{(N-1)}{(1-s)} + \frac{2L_h P}{K_h}\frac{s}{(1-s)^2}.$$

On setting $A_1 = (2b-1) A_o - \frac{L_h P}{K_h}(N - 1/2)$, dividing through by the

coefficient of $\bar{A}$, and using partial fractions, this becomes;

$$\bar{A} = \frac{A_o(1-bs)}{1-2bs+s^2} + \frac{(b-1)A_o s}{1-2bs+s^2} + \frac{L_h P}{3K_v}\left[\frac{N}{1-s} - \frac{(1-bs)N - 1/2\,(b+1)s}{1-2bs+s^2} - \frac{s}{(1-s)^2}\right].$$

The inverse transform is found to be;

$$A_n = A_o\left[\cosh\lambda n + (b-1)\frac{\sinh\lambda n}{\sinh\lambda}\right] + \frac{L_h P}{3K_v}\left[(N-n) - N\cosh\lambda n + \frac{b+1}{2}\frac{\sinh\lambda n}{\sinh\lambda}\right].$$

Since $A_N = 0$, then

$$A_o = \frac{L_h P}{3K_v}\left[\frac{N\cosh\lambda N\sinh\lambda - 1/2\,(b+1)\sinh\lambda N}{\cosh\lambda N\sinh\lambda + (b-1)\sinh\lambda N}\right], \text{ and}$$

$$A_n = \frac{L_h P}{3K_v}\left[\frac{N\cosh\lambda N\sinh\lambda - 1/2\,(b+1)\sinh\lambda N}{\cosh\lambda N\sinh\lambda + (b-1)\sinh\lambda N}\right]\left[\cosh\lambda n + (b-1)\frac{\sinh\lambda n}{\sinh\lambda}\right]$$

$$+ \frac{L_h P}{3K_v}\left[(N-n) - N\cosh\lambda n + 1/2\,(b+1)\frac{\sinh\lambda n}{\sinh\lambda}\right]$$

$$= \frac{L_h P}{3K_v}\left[(N-n) - \frac{\frac{b+1}{2} + (b-1)N}{\sinh\lambda(N+1) - \sinh\lambda N}\sinh\lambda(N-n)\right]$$

where $\cosh\lambda = b = 1 + 3\frac{K_v}{K_h}$.

5. Consider a set of n discs of moment of Inertia I connected by a massless shaft of stiffness c. Assume oscillation to be harmonic. Then the difference equation where θ_x is the amplitude of the oscillation of the x'th disc, is $\theta_{x+2} - 2\left(1 - \frac{Iw^2}{2c}\right)\theta_{x+1} + \theta_x = 0$ where $\theta_o = \theta_1$ and $\theta_n = \theta_{n+1}$

The transformed equation is

$$\left(\frac{1}{s^2} - \frac{2b}{s} + 1\right)\bar{\theta} = \frac{\theta_o}{s^2} + \frac{\theta_1}{s} - \frac{2b\theta_o}{s}, \text{ where } b = 1 - \frac{Iw^2}{2c}$$

therefore $\bar{\theta} = \theta_o\frac{1+(1-2b)s}{1-2bs+s^2} = \theta_o\frac{1-bs+(1-b)s}{1-2bs+s^2}$.

The result of taking the inverse transformation is

$$\theta_x = \theta_o\left[\cos\lambda x + (1-\cos\lambda)\frac{\sin\lambda x}{\sin\lambda}\right], \left(\text{where } \cos\lambda = b\right)$$

$$= \frac{\theta_o}{\cos\frac{\lambda}{2}}\left[\cos\lambda x\cos\frac{\lambda}{2} + \sin\lambda x\sin\frac{\lambda}{2}\right]$$

$$= \theta_o \frac{\cos\lambda(x - 1/2)}{\cos\lambda/2}$$

Since $\theta_{n+1} = \theta_n$, then $\cos\lambda(n - 1/2) = \cos\lambda(n + 1/2)$ and therefore $\lambda = \frac{r\pi}{n}$ where $r = 0, 1, - - -$. Hence $\theta_x = \theta_o \frac{\cos\frac{r\pi}{n}(x - 1/2)}{\cos\frac{r\pi}{2n}}$

The natural frequencies of the system can be obtained by solving $1 - \frac{Iw_r^2}{2c} = \cos\frac{r\pi}{n}$ for w_r.

The application of the method to simultaneous equations may be illustrated by considering the following problem.

6. Two perfectly elastic spheres of masses M and m, M > m lie on a smooth horizontal surface with their line of centers perpendicular to a smooth perfectly elastic wall. The sphere of mass M is projected towards the wall so as to impinge directly with velocity V on the sphere of mass m. Find the velocities of the spheres after the x'th impact.

Let u_x and v_x be the velocities of M and m respectively after the x'th impact, measured positive when toward the wall, then[5]

$$M(u_{x+1} - u_x) + m(v_{x+1} + v_x) = 0$$

and $u_{x+1} + u_x - (v_{x+1} - v_x) = 0$ where $u_o = V$ and $v_o = 0$.

The transformed equations are

$$M\left(\frac{1}{s} - 1\right)\bar{u} + m\left(\frac{1}{s} + 1\right)\bar{v} = \frac{MV}{s}$$

and $$\left(\frac{1}{s} + 1\right)\bar{u} - \left(\frac{1}{s} - 1\right)\bar{v} = \frac{V}{s}$$

or $$M(1-s)\bar{u} + m(1+s)\bar{v} = MV$$

and $$(1+s)\bar{u} - (1-s)\bar{v} = V$$

Hence $$u = \frac{\begin{vmatrix} MV & m(1+s) \\ V & -(1-s) \end{vmatrix}}{\begin{vmatrix} M(1-s) & m(1+s) \\ 1+s & -(1-s) \end{vmatrix}} = \frac{V\left[(M+m)-(M-m)s\right]}{M+m-2(M-m)s+(M+m)s^2} = \frac{V\left[1 - \frac{M-m}{M+m}s\right]}{1 - 2\frac{M-m}{M+m}s+s^2}$$

and $\bar{v} = \dfrac{2MVs}{M+m-2(M-m)s+(M+m)s^2} = \dfrac{2MV}{M+m} \dfrac{s}{1-2\frac{M-m}{M+m}s+s^2}$

The result of taking the inverse transforms is

$$u_x = V \cos \lambda x \quad ,$$

and $v_x = \dfrac{2MV}{M+m} \dfrac{\sin \lambda x}{\sin \lambda} = \sqrt{\dfrac{M}{m}} V \sin \lambda x$

where $\cos \lambda = \dfrac{M-m}{M+m}$

The method may also be used to solve linear difference equations with linear coefficients as is illustrated in the following problem.

7. Find y_k where $(k+2)\, y_{k+2} - (k+3)\, y_{k+1} + y_k = 0$.

The transformed equation is $\dfrac{1}{s}\dfrac{d\bar{y}}{ds} - \dfrac{d\bar{y}}{ds} - \dfrac{2\bar{y}}{s} + \bar{y} = \dfrac{y_1}{s} - \dfrac{2y_o}{s}$

or $\dfrac{d\bar{y}}{ds} - \dfrac{2-s}{1-s}\bar{y} = \dfrac{y_1-2y_o}{1-s}$ which is a linear differential equation of the first order and first degree whose solution is

$$\bar{y} = -\frac{(y_1-2y_o)}{1-s} + \frac{C\, e^s}{1-s}$$

The result of taking the inverse transform is

$$y_k = -(y_1 - 2y_o) + C \sum_{n=0}^{k} \frac{1}{n!}$$

But when $k = 0$

$$y_o = -(y_1 - 2y_o) + C, \text{ therefore } C = y_1 - y_o$$

Hence $y_k = -(y_1 - 2y_o) + (y_1 - y_o) \sum_{n=0}^{k} \dfrac{1}{n!}$

APPENDIX I

A Short Table of Formulas for the Power Series Transform

Sequence	Transform
1. $\{f_k\}$	$\bar{f}(s) = \sum_0^\infty f_k s^k$
2. $\{f_{k+n}\}$	$\frac{\bar{f}}{s^n} - \frac{1}{s^n} \sum_0^{n-1} f_k x^k$
3. $\{f_{k-n}\}$	$s^n \bar{f}$, where $f_{-k} = 0$
4. $\{k f_k\}$	$s \frac{d\bar{f}}{ds}$
5. $\{(k+1) f_{k+1}\}$	$\frac{d\bar{f}}{ds}$
6. $\{(k+2) f_{k+2}\}$	$\frac{1}{s} \frac{d\bar{f}}{ds} - \frac{f_1}{s}$
7. $f_k = 1$	$\frac{1}{1-s}$
8. $f_k = r^k$	$\frac{1}{1-rs}$
9. $f_k = k$	$\frac{s}{(1-s)^2}$
10. $f_k = \frac{k(k+1)}{2}$	$\frac{s}{(1-s)^3}$
11. $f_k = \frac{k(k+1)(k+2)}{3!}$	$\frac{s}{(1-s)^4}$
12. $f_k = k r^{k-1}$	$\frac{s}{(1-rs)^2}$

Sequence	Transform
13. $f_k = \frac{r_1^k - r_2^k}{r_1 - r_2}$	$\frac{s}{1+2bs+cs^2}$, $b^2 - c > 0$
14. $f_k = c^{\frac{k-1}{2}} \frac{\sin k\lambda}{\sin \lambda}$, where $\cos\lambda = -\frac{b}{\sqrt{c}}$	$\frac{s}{1+2\,bs+cs^2}$, $b^2 - c < 0$
15. $f_k = c^{\frac{k}{2}} \cos k\lambda$, $\cos\lambda = -\frac{b}{\sqrt{c}}$	$\frac{1 + b\,s}{1 + 2bs + cs^2}$ $b^2 - c < 0$
16. $f_k = \begin{cases} \frac{\sinh k\lambda}{\sinh \lambda}, & \cosh\lambda = -b \\ (-1)^{k-1} \frac{\sinh k\lambda}{\sinh \lambda}, & \cosh\lambda = b \end{cases}$	$\frac{s}{1+2\,bs + s^2}$, $b^2 - 1 > 0$
17. $f_k = \frac{\sin k\lambda}{\sin \lambda}$, $\cos\lambda = -b$	$\frac{s}{1+ 2bs + s^2}$, $b^2 - 1 < 0$
18. $f_k = \begin{cases} \cosh k\lambda, & \cosh\lambda = -b \\ (-1)^k \cosh k\lambda, & \cosh\lambda = b \end{cases}$	$\frac{1 + b\,s}{1 + 2bs + s^2}$, $b^2 - 1 > 0$
19. $f_k = \cos k\lambda$, $\cos\lambda = -b$	$\frac{1 + b\,s}{1+ 2bs + s^2}$, $b^2 - 1 < 0$
20. $f_k = \sum_{i=0}^{k} g_{k-i}\, h_i = g_k^* h_k$	$\bar{f} = \bar{g} \cdot \bar{h}$
21. $F(k) = \sum_{i=0}^{k} f_i$	$\bar{F} = \frac{\bar{f}}{1-s}$

APPENDIX II

Proofs of Some of the Formulas in Appendix I

Proof of 2. Since $\bar{f} = f_o + f_1 s + f_2 s^2 + - - - + f_k s^k + - - -$

then $\frac{\bar{f}}{s^n} - \frac{1}{s^n}(f_o + f_1 s + - - - + f_{n-1} s^{n-1})$

$= f_n + f_{n+1} s + f_{n+2} s^2 + - - - + f_{n+k} s^k + - - -$

$= \sum_0^\infty f_{n+k} s^k$

Proof of 4. Since $\frac{d\bar{f}}{ds} = f_1 + 2 f_2 s + - - - + k f_k s^{k-1} + - - -$

then $s\frac{d\bar{f}}{ds} = f_1 s + 2 f_2 s^2 + - - - + k f_k s^k + - - -$

$= \sum_0^\infty k f_k s^k$

Proof of 8. $\frac{1}{1-rs} = (1-rs)^{-1} = 1 + rs + r^2 s^2 + - - + r^k s^k + - - -$

$= \sum_0^\infty r^k s^k$

Proof of 13. $\frac{s}{1+2bs+cs^2} = \frac{s}{(1-r_1 s)(1-r_2 2)}$ where $r_1 = -b + \sqrt{b^2-c}$ and $r_2 = -b - \sqrt{b^2-c}$

$= \frac{1}{r_1 - r_2}\left(\frac{1}{1-r_1 s} - \frac{1}{1-r_2 s}\right)$

$= \frac{1}{r_1 - r_2}\left(\sum_0^\infty r_1^k s^k - \sum_0^\infty r_2^k s^k\right)$ (by 8)

$= \sum_0^\infty \frac{r_1^k - r_2^k}{r_1 - r_2} s^k$

Proof of 14. If $b^2 - c < 0$ then r_1 and r_2 are complex conjugates and if $r_1 = p + i q$ then $r_2 = \frac{p^2 + q^2}{p + i q} = \frac{c}{p + i q}$.

Hence $$\frac{r_1^k - r_2^k}{r_1 - r_2} = \frac{(p+iq)^k - (\frac{p+iq}{c})^{-k}}{p+iq - (\frac{p+iq}{c})^{-1}} = \frac{c^{\frac{k}{2}}\left[(\frac{p+iq}{c^{1/2}})^k - (\frac{p+iq}{c^{1/2}})^{-k}\right]}{c^{1/2}\left[\frac{p+iq}{c^{1/2}} - (\frac{p+iq}{c^{1/2}})^{-1}\right]}$$

$$= c^{\frac{k-1}{2}} \frac{\sin \lambda k}{\sin \lambda}$$

where $i\lambda = \log \frac{p+iq}{c^{1/2}} = \log \sqrt{\frac{p^2+q^2}{c}} + i \arctan \frac{q}{p} = i \arctan \frac{q}{p}$.

Thus $\lambda = \arctan \frac{q}{p} = \arccos \frac{p}{\sqrt{p^2+q^2}} = \arccos \frac{-b}{\sqrt{c}}$.

Proof of 15. Note that $\frac{\sin(k+1)\lambda}{\sin \lambda} = \frac{\sin k\lambda \cos \lambda + \cos k\lambda \sin \lambda}{\sin \lambda}$,

hence $\cos k\lambda = \frac{\sin(k+1)\lambda - \sin k\lambda \cos \lambda}{\sin \lambda}$.

Now from 14) and 3), $P\left(c^{\frac{k}{2}} \frac{\sin(k+1)\lambda}{\sin \lambda}\right) = \frac{1}{1 + 2bs + c\,s^2}$

and from 14), $P\left(c^{\frac{k-1}{2}} \frac{\sin k\lambda}{\sin \lambda}\right) = \frac{s}{1 + 2bs + c\,s^2}$

Then $P\left(c^{\frac{k}{2}} \cos k\lambda\right) = \frac{1 - sc^{1/2} \cos \lambda}{1 + 2bs + c\,s^2} = \frac{1 + bs}{1 + 2bs + c\,s^2}$.

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